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The Polynomial Solutions of the Difference Equation

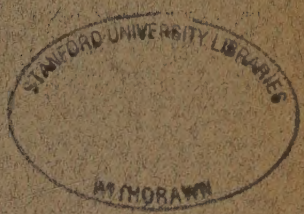
$$af(x+1) + bf(x) = \Phi(x)$$

A Thesis

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF CORNELL UNIVERSITY IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

BY



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September, 1925

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The polynomial solutions of the difference equation $af(x+1)+bf(x)=\varphi(x)$.

Extracted from
THE TÔHOKU MATHEMATICAL JOURNAL, Vol. 39, Pt. 2,
edited by T. HAYASHI, M. FUJIWARA, T. KUBOTA, Y. OKADA, and
T. TAKASU, College of Science, Tôhoku Imperial University,
Sendai, Japan.

June 1934

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The Polynomial Solution of the Difference Equation $af(x+1)+bf(x)=\varphi(x)$ [1],

by

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Introduction.

The numbers of Bernoulli were introduced into mathematical analysis more than two centuries ago. However, it was not until a much later period that they were found to be closely connected with such difference equations as $f(x+1)-f(x)=\nu x^{\nu-1}$. This equation and the related equation $f(x+1)+f(x)=2x^{\nu}$ have since become the center of detailed investigation.

In the present paper we shall study polynomial solutions of a more general difference equation

$$af(x+1)+bf(x)=\varphi(x),$$

where a, b and $a+b$ are complex numbers different from zero and $\varphi(x)$ is a given polynomial, and, in particular, the equation

$$af(x+1)+bf(x)=(a+b)x^{\nu},$$

related equations, and sets of numbers defined by certain recursion formulae and useful in the solutions of these equations and the summation of certain power series.

The very elegant methods used by N. E. Nörlund[2] in an investigation of the equations $f(x+1)-f(x)=\nu x^{\nu-1}$, $f(x+1)+f(x)=2x^{\nu}$ are the ones adopted in connection with the equations which form the subject of the present study. It is interesting to note that the symbolic methods which Lucas and Blissard introduced into the theory of Bernoulli numbers lend themselves admirably to this investigation.

In the course of the preparation of this paper there appeared Nörlund's treatise on the difference calculus, entitled '*Differenzenrechnung*', chapters one, two, and six of which include the contents of '*Les Polynomes de Bernoulli*'. On pages 398-400 polynomial

[1] An abridgment of a thesis presented to the Faculty of the Graduate School of Cornell University for the degree of Doctor of Philosophy, September 5th, 1925.

[2] *Memoire sur les polynomes de Bernoulli*, Acta Mathematica, vol. 43 (1920), p. 121. Hereafter in this paper this memoir will be referred to as *Les Polynomes de Bernoulli*.

solutions of difference equations of the same type as studied in this paper are discussed.

Birkhoff([1]) has proved that the general equation, when $\varphi(x) = \rho x^{\nu}$, has, for every positive integral ν, a unique polynomial solution, while E. W. Barnes([2]) and E. Picard([3]) have investigated the general equation where $\varphi(x)$ is an analytic function.

In part I of the present paper we consider properties of what we call the *generalized Euler polynomial* which is the polynomial solution of the general equation where $\varphi(x) = (a+b)x^{\nu}$. We discuss certain special cases of this same equation in part II and, finally, in part III, *generalized Euler polynomials* of higher order are considered.

Part 1. The Generalized Euler Polynomial.

1. The definition of the *generalized Euler polynomial.* For the purposes of this paper we shall use certain sequences of numbers defined by symbolical formulae. One such sequence comprises the numbers K defined by

$$(1) \qquad a(K+a+b)^{\nu}+bK^{\nu}=0, \ \nu=1, 2, 3, \ldots, \ K_0=1,$$

where in the binomial expansion K^r is to be replaced by K_r. Thus putting $\nu=1, 2, 3, \ldots$, in succession, we obtain

$$K_0=1, \ K_1=-a, \ K_2=a(a-b), \ K_3=-a(a^2-4ab+b^2), \ \ldots.$$

If we put $\dfrac{K}{a+b}=S$ in (1) and after expansion replace S^r by S_r so that $S_r=\dfrac{K_r}{(a+b)^r}$, (1) becomes

$$(2) \qquad a(S+1)^{\nu}+bS^{\nu}=0, \ \nu=1, 2, 3, \ldots, \ S_0=1.$$

Now, if we let $\varphi(z)$ be any polynomial in z we have from (2) the symbolic relation

$$a\varphi(S+1)+b\varphi(S)=(a+b)\varphi(0)$$

from which, upon replacing $\varphi(z)$ by $\varphi(z+x)$, we obtain

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([1]) *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43 (1922), p. 5.

([2]) *On functions generated by linear difference equations of the first order*, Proc. London Math. Soc., series 2, vol. II. (1904), p. 288.

([3]) *Sur une classe de transcendantes nouvelles*, Comptes Rendus, vol. 117 (1893), p. 472; Acta Mathematica, vol. 18 (1894), p. 133.

$$a\varphi(x+S+1)+b\varphi(x+S)=(a+b)\varphi(x),$$

so that the symbolic expansion of $\varphi(x+S)$ is a solution of the difference equation

$$af(x+1)+bf(x)=(a+b)\varphi(x), \tag{3}$$

whence

$$f(x)=\varphi(x+S)=\varphi(x)+\frac{S_1}{1!}\varphi'(x)+\frac{S_2}{2!}\varphi''(x)+\frac{S_3}{3!}\varphi'''(x)+\ \ldots. \tag{4}$$

If we define the operation $a\underset{\omega}{\nabla}b$ by

$$a\underset{\omega}{\nabla}bf(x)=\frac{af(x+\omega)+bf(x)}{a+b} \tag{5}$$

and similarly $a\nabla b$ by

$$a\nabla bf(x)=\frac{af(x+1)+bf(x)}{a+b}, \tag{5a}$$

the relation (3) then becomes

$$a\nabla bf(x)=\varphi(x). \tag{6}$$

Now, letting $\varphi(x)$ have the special value x^ν and denoting the corresponding polynomial in (4) by $U_\nu(x)$, from (6) we obtain

$$a\nabla bU_\nu(x)=x^\nu. \tag{7}$$

The *generalized Euler polynomial* is defined by

$$U_\nu(x)=(x+S)^\nu=x^\nu+\nu S_1x^{\nu-1}+\frac{\nu(\nu-1)}{2!}S_2x^{\nu-2}+\ldots.+S_\nu. \tag{8}$$

2. The *generalized Boole summation formula.* Since from (8), $U_\nu(x+h)=(x+S+h)^\nu$, then (4) gives

$$f(x+h)=\varphi\{x+U(h)\}=\sum_{s=0}^{\nu}\varphi^{(s)}(x)\frac{U_s(h)}{s!}. \tag{9}$$

If the operator $a\nabla b$ is applied to the right member of (9), the following generalization of the Boole summation formula(1) is obtained:

$$\varphi(x+h)=a\nabla b\varphi(x)+U_1(h)a\nabla b\varphi'(x)+\frac{U_2(h)}{2!}a\nabla b\varphi''(x)+\ldots.. \tag{10}$$

We now give a very simple application of (10). If, in (10), we put $b=-1$ and $a=k$, where $k\neq1$, we obtain

$$\varphi(x+h)=\frac{1}{k-1}\Big\{[k\varphi(x+1)-\varphi(x)]+\frac{U_1(h)}{1!}[k\varphi'(x+1)-\varphi'(x)]$$
$$+\frac{U_2(h)}{2!}[k\varphi''(x+1)-\varphi''(x)]+\ldots.\Big\}. \tag{11}$$

(1) Cf.: Nörlund, *Les Polynomes de Bernoulli*, p. 135.

Now, if we add member to member the equations obtained by giving x in (11) the values $0, 1, 2, 3, \ldots, n$ in succession and multiplying through by the numbers $1, k, k^2, \ldots, k^n$, respectively, we get

$$(12)\quad \sum_{r=0}^{n} k^r \varphi(r+h) = \frac{1}{k-1}\Big\{[k^{n+1}\varphi(n+1)-\varphi(0)] + \frac{U_1(h)}{1!}[k^{n+1}\varphi'(n+1)-\varphi'(0)] + \frac{U_2(h)}{2!}[k^{n+1}\varphi''(n+1)-\varphi''(0)]+\ldots.\Big\}.$$

Putting $h=0$ and noting from (8) that $U_\nu(0)=S_\nu$, (12) becomes

$$(13)\quad \sum_{r=0}^{n} k^r \varphi(r) = \frac{1}{k-1}\Big\{[k^{n+1}\varphi(n+1)-\varphi(0)] + \frac{S_1}{1!}[k^{n+1}\varphi'(n+1)-\varphi'(0)] + \frac{S_2}{2!}[k^{n+1}\varphi''(n+1)-\varphi''(0)]+\ldots.\Big\}.$$

Formula (13) may be used in summing certain finite series. We further note that the first member of (13) is the sum of the first n terms of a certain power series in k. The equation (13) therefore offers a means of summing series of this type. Now, if we consider the case when $|k|<1$ and let n become infinite, we find

$$(14)\quad \operatorname*{Lim}_{n\to\infty} \sum_{r=0}^{n} k^r \varphi(r) = \frac{1}{1-k}\varphi(S), \quad |k|<1,$$

where, in the expansion of the second member, S^i is to be replaced by S_i.

3. Some properties of the *generalized Euler polynomial.* If we return now to equation (3) and use (9), we find

$$(15)\quad a\varphi\{U(x)+1\}+b\varphi\{U(x)\}=(a+b)\varphi(x).$$

Then, putting $\varphi(x)=x^\nu$, we have

$$(16)\quad a[U(x)+1]^\nu+b[U(x)]^\nu=(a+b)x^\nu,$$

which, when expanded symbolically, gives

$$(17)\quad (a+b)U_\nu(x)+a\binom{\nu}{1}U_{\nu-1}(x)+a\binom{\nu}{2}U_{\nu-2}(x)+\ldots.+aU_0(x) = (a+b)x^\nu.$$

Giving ν the values $0, 1, 2, \ldots.$ in succession, we have

$$U_0(x)=1,\ U_1(x)=x-\frac{a}{a+b},\ U_2(x)=x^2-\frac{2a}{a+b}x+a\frac{a-b}{(a+b)^2},\ \ldots.$$

Further, it becomes evident, upon replacing x by $-x$ in equation (7), that the equation, $b\nabla af(x)=x^\nu$ is satisfied by $f(x)=(-1)^\nu U_\nu(1-x)$.

It is easily verified that $f(x)=\sum_{s=0}^{n-1} G_s U_\nu\left(\frac{x+s}{n}\right)$ satisfies (7), if $G_s=n^\nu(-1)^s\left(\frac{a}{b}\right)^s$, where $-\frac{a}{b}$ is one of the $(n-1)$th roots of unity, that is, (7) is satisfied by

$$f(x)=n^\nu\sum_{s=0}^{n-1}(-1)^s\left(\frac{a}{b}\right)^s U_\nu\left(\frac{x+s}{n}\right),$$

which can be verified by substitution in (7). As the polynomial solution of (7) is unique, we have, replacing x by nx,

$$U_\nu(nx)=n^\nu\sum_{s=0}^{n-1}(-1)^s\left(\frac{a}{b}\right)^s U_\nu\left(x+\frac{s}{n}\right), \tag{18}$$

where $\left(-\frac{a}{b}\right)^{n-1}=1$.

The polynomials $U_\nu(x)$ can be expressed by the aid of the *generalized numbers of Euler* U_ν defined by the symbolical relation

$$a(U+1)^\nu+b(U-1)^\nu=0,\ \nu=1, 2, 3, \ldots,\ U_0=1. \tag{19}$$

The values of the first U's are

$$U_0=1,\ U_1=-\frac{a-b}{a+b},\ U_2=\frac{a^2-6ab+b^2}{(a+b)^2}.$$

Let us now turn to the difference equation

$$aF_\nu\left(x+\frac{2}{a+b}\right)+bF_\nu(x)=(a+b)x^\nu, \tag{20}$$

which is satisfied by

$$F_\nu(x)=\left(\frac{2}{a+b}\right)^\nu U_\nu\left(\frac{x}{\frac{2}{a+b}}\right). \tag{21}$$

Put $F_\nu(x)=\sum_{s=0}^{\nu}\binom{\nu}{s}A_s\left(x-\frac{1}{a+b}\right)^{\nu-s}$, substitute in (20), equate the coefficients of like powers of x, and we obtain

$$a\sum_{s=0}^{\nu}\binom{\nu}{s}(a+b)^{\nu-s}A_{\nu-s}+b\sum_{s=0}^{\nu}(-1)^s\binom{\nu}{s}(a+b)^{\nu-s}A_{\nu-s}=0, \tag{22}$$

$$\nu=1, 2, 3, \ldots,\ A_0=1.$$

Comparing (22) with (19), we see that $U_\nu=(a+b)^\nu A_\nu$. Consequently we have

$$F_\nu(x)=\sum_{s=0}^{\nu}\binom{\nu}{s}\frac{U_s}{(a+b)^s}\left(x-\frac{1}{a+b}\right)^{\nu-s}. \tag{23}$$

Putting $x=\frac{1}{a+b}$ in (21) and (23) we get

$$F_\nu\left(\frac{1}{a+b}\right)=\left(\frac{2}{a+b}\right)^\nu U_\nu\left(\frac{1}{2}\right)$$
$$=\frac{U_\nu}{(a+b)^\nu}.$$

Whence

$$U_\nu\left(\frac{1}{2}\right)=\frac{U_\nu}{2^\nu},$$

a relation already proved by Nörlund in the special case $a=b=1$([1]).

Again, if $\varphi(z)$ be any polynomial, from (19) we obtain $a\varphi(U+1)+b\varphi(U-1)=(a+b)\varphi(0)$ which gives, upon replacing $\varphi(z)$ by $\varphi\left(x+\frac{z}{a+b}\right)$,

$$(24)\qquad a\varphi\left(x+\frac{U+1}{a+b}\right)+b\varphi\left(x+\frac{U-1}{a+b}\right)=(a+b)\varphi(x),$$

whence the following is true:

The difference equation $af\left(x+\frac{2}{a+b}\right)+bf(x)=(a+b)\varphi(x)$, *where* $\varphi(x)$ *is any polynomial of degree* ν, *has a solution*

$$f(x)=\varphi\left(x+\frac{U-1}{a+b}\right)=\sum_{s=0}^{\nu}\varphi^{(s)}\left(x-\frac{1}{a+b}\right)\frac{1}{(a+b)^s}\frac{U_s}{s!}.$$

In the special case where $\varphi(x)=x^\nu$, $f(x)$ becomes $F(x)$ and we have (21) and (23) which, upon replacing x by $x+h$, give

$$(25)\qquad U_\nu(x+h)=\left(x+\frac{U-1}{2}+h\right)^\nu.$$

Expanding the left member of (25), we obtain

$$(26)\quad U_\nu(x+h)=U_\nu(x)+\binom{\nu}{1}hU_{\nu-1}(x)+\binom{\nu}{2}h^2U_{\nu-2}(x)+\ldots.$$
$$+h^\nu U_0(x),$$

which reduces to (17), if $h=1$.

The polynomial defined by the difference equation

$$(27)\qquad af\left(x+h+\frac{2}{a+b}\right)+bf(x+h)=(a+b)\varphi(x+h)$$

can be developed in the following manner:

$$(28)\qquad f(x+h)=\varphi\{x+F(h)\}$$
$$=\varphi\left\{x+\left(\frac{2}{a+b}\right)^\nu U\left(\frac{h}{\frac{2}{a+b}}\right)\right\}$$

([1]) Cf.: Nörlund, loc. cit., p. 133.

$$=\sum_{s=0}^{\nu}\varphi^{(s)}(x)\left(\frac{2}{a+b}\right)^{s}\frac{U_s\left(\frac{h}{2/(a+b)}\right)}{s\,!}.$$

It is readily verified that the last member of (28) satisfies (27), by applying to the first and last members of (28) the operation $a\underset{\frac{2}{a+b}}{\nabla}b$ taken with respect to h.

We may consider (24) as a recursion formula for $U_\nu(x)$. We can write (24) in the form

$$(29)\quad a\varphi\left\{\left(\frac{2}{a+b}\right)U\left(\frac{x}{\frac{2}{a+b}}\right)+\frac{2}{a+b}\right\}+b\varphi\left\{\left(\frac{2}{a+b}\right)U\left(\frac{x}{\frac{2}{a+b}}\right)\right\}=(a+b)\,\varphi(x).$$

If we place $\varphi(x)=\dfrac{x^\nu}{\left(\frac{2}{a+b}\right)^\nu}$, we obtain formula (16). Putting, in particular, $x=\dfrac{1}{2}$, we get

$$(30)\qquad a\sum_{s=0}^{\nu}\binom{\nu}{s}2^s U_{\nu-s}+b\,U_\nu=a+b.$$

Returning now to the numbers K, defined at the beginning of this paper, we make use of them in the solutions of certain difference equations.

If $\varphi(z)$ be any polynomial, we have from equation (1) the relation $a\varphi(K+a+b)+b\varphi(K)=(a+b)\,\varphi(0)$ which, upon replacing $\varphi(z)$ by $\varphi\left(x+\dfrac{z}{a+b}\right)$, becomes

$$(31)\qquad a\varphi\left(x+1+\frac{K}{a+b}\right)+b\varphi\left(x+\frac{K}{a+b}\right)=(a+b)\,\varphi(x).$$

We have the following:

The difference equation $a\nabla b\,f(x)=\varphi(x)$, $\varphi(x)$ *being of degree* ν, *has the unique polynomial solution*

$$f(x)=\varphi\left(x+\frac{K}{a+b}\right)=\sum_{s=0}^{\nu}\frac{\varphi^{(s)}(x)}{s\,!}\,\frac{K}{(a+b)^s}.$$

If we place $\varphi(x)=x^\nu$, we have from the definition of the *generalized Euler polynomial*

$$(32)\qquad U_\nu(x)=\left(x+\frac{K}{a+b}\right)^\nu=\sum_{s=0}^{\nu}\binom{\nu}{s}(a+b)^{-s}K_s x^{\nu-s}.$$

Now, in equation (31) put $\varphi(x)=x^\nu$. We get then

$$a(x+K+a+b)^\nu+b(x+K)^\nu=(a+b)x^\nu,$$

which becomes

$$a\left(K+\frac{a+b}{2}\right)^\nu+b\left(K-\frac{a+b}{2}\right)^\nu=(a+b)(-1)^\nu\left(\frac{a+b}{2}\right)^\nu,$$

when $x=-\frac{a+b}{2}$, thus giving, when $a=b=1$, according as ν is even or odd,

$$(K+1)^\nu+(K-1)^\nu=\pm 2. \tag{33}$$

When we put $x=\frac{1}{a+b}$ in (32), we obtain

$$(a+b)^\nu U_\nu\left(\frac{1}{a+b}\right)=(K+1)^\nu$$

$$=\sum_{s=0}^{\nu}\binom{\nu}{s}K_s,$$

which gives, putting $a=b=1$ and substituting E for U and C for K,

$$E_\nu\left(\frac{1}{2}\right)=\frac{(C+1)^\nu}{2^\nu},$$

or, since $E_\nu\left(\frac{1}{2}\right)=\frac{E_\nu}{2^\nu}$,

$$E_\nu=(C+1)^\nu\,(^1).$$

If we put $x=0$ in (32), we have $U_\nu(0)=\left(\frac{K}{a+b}\right)^\nu$ which gives, when $a=b=1$, and U is replaced by E and K by C,

$$E_\nu(0)=\frac{C_\nu}{2^\nu}\,(^2).$$

The E's and C's are numbers defined by Nörlund(3), the E's having been previously given in this section.

Part II. Certain Special Difference Equation and Number Sequences Connected with the Difference Equation $af(x+1)+bf(x)=\varphi(x)$.

4. Some properties of the polynomials, $B_\nu(x)$, $C_\nu(x)$, $D_\nu(x)$ and

(1) Cf.: Nörlund, *Les Polynomes de Bernoulli*, p. 138.
(2) Cf.: Nörlund, loc. cit., p. 138.
(3) Loc. cit., pp. 134 and 136, respectively.

$E_\nu(x)$. We now consider the equations

(34) $f(x+1)-f(x)=\nu x^{\nu-1}$, $\nu=1, 2, 3, \ldots$,

(35) $f(x+1)-if(x)=\nu x^{\nu-1}$, $\nu=1, 2, 3, \ldots$, $i^2=-1$,

(36) $f(x+1)+if(x)=x^\nu$, $\nu=0, 1, 2, \ldots$, $i^2=-1$,

(37) $f(x+1)+f(x)=2x^\nu$, $\nu=0, 1, 2, \ldots$.

The following theorem can be readily proved:

Equations (35), (36), *and* (37) *have, respectively, for every value of* ν *indicated, a unique polynomial solution, while equation* (34) *has, for every value of* ν *indicated, an infinite number of polynomial solutions, any two such solutions, for a given* ν, *differing by a constant.*

Let $B_\nu(x)$ be the polynomial solution of (34) satisfying the condition $B_\nu(0)=B_\nu$, where B_ν is the Bernoulli number given symbolically by $(B+1)^\nu-B^\nu=0$, $\nu=2, 3, 4, \ldots$, $B_0=1$([1]). Also denote the polynomial solutions of (35), (36), and (37) by $C_\nu(x)$, $D_\nu(x)$ and $E_\nu(x)$([2]), respectively.

Rewriting equation (36) in the form, $D_\nu(x+1)+iD_\nu(x)=x^\nu$ and then, replacing x by $-nx$, we see that $f(x)=i(-1)^\nu n^{-\nu}D_\nu(1-nx)$ satisfies the difference equation $f\left(x+\frac{1}{n}\right)-if(x)=x^\nu$. This latter difference equation is further satisfied by $f(x)=\sum_{s=0}^{n-1} i^{3s+2}D_\nu\left(x+\frac{s}{n}\right)$, if n is an integer of the form $4k+3$. Hence we have the following relation:

(38) $D_\nu(1-nx)=n^\nu(-1)^\nu\sum_{s=0}^{n-1} i^{3s+1}D_\nu\left(x+\frac{s}{n}\right)$, $n=4k+3$.

In a similar manner we obtain the three following equations:

(39) $D_\nu(1-nx)=n^\nu(-1)^\nu\sum_{s=0}^{n-1} i^{3s+2}\frac{B_{\nu+1}\left(x+\frac{s}{n}\right)}{\nu+1}$, $n=4k$;

(40) $D_\nu(1-nx)=n^\nu(-1)^\nu\sum_{s=0}^{n-1} i^{3s+3}\frac{C_{\nu+1}\left(x+\frac{s}{n}\right)}{\nu+1}$, $n=4k+1$;

(41) $D_\nu(1-nx)=n^\nu(-1)^\nu\sum_{s=0}^{n-1} i^{3s}\frac{E_\nu\left(x+\frac{s}{n}\right)}{2}$, $n=4k+2$.

Now, if in (40) we put $n=1$, we get

([1]) Cf.: Nörlund, *Les Polynomes de Bernoulli*, pp. 123, 124.

([2]) Cf.: Nörlund, loc. cit., p. 131.

$$D_\nu(1-x)=(-1)^{\nu+1}i\frac{C_{\nu+1}(x)}{\nu+1}. \tag{42}$$

Next, upon comparing (35) with (36), we note that $D_\nu(x)$ and $\frac{C_{\nu+1}(x)}{\nu+1}$ are conjugate. Hence (42) may be written in the form $D_\nu(1-x)=(-1)^{\nu+1}i\overline{D}_\nu(x)$, where $\overline{D}_\nu(x)$ is the conjugate of $D_\nu(x)$. Also we obtain readily the relation $C_\nu(1-x)=(-1)^{\nu+1}\overline{C}(x)$.

Again, if in (38), (39), (40) and (41) we put $x=\frac{1-y}{n}$, we obtain other expressions for $D_\nu(y)$. If the immediately preceding substitution be used for x in (41) and y be then replaced by x, we have

$$D_\nu(x)=\frac{n^\nu}{2}(-1)^\nu\sum_{s=0}^{n-1}i^{3s}E_\nu\left(\frac{1+s-x}{\nu}\right),\quad n=4k+2. \tag{43}$$

Putting $n=2$ in (43), we find

$$D_\nu(x)=2^{\nu-1}(-1)^\nu\left[E_\nu\left(\frac{1-x}{2}\right)-iE_\nu\left(\frac{2-x}{2}\right)\right]. \tag{44}$$

Then, replacing x by $1-x$ and using relation (42), we have

$$C_{\nu+1}(x)=(\nu+1)2^{\nu-1}\left[E_\nu\left(\frac{x+1}{2}\right)+iE_\nu\left(\frac{x}{2}\right)\right]. \tag{45}$$

Again, since $D_\nu(x)$ and $\frac{C_{\nu+1}(x)}{\nu+1}$ are conjugates, from (45) we obtain

$$D_\nu(x)=2^{\nu-1}\left[E_\nu\left(\frac{x+1}{2}\right)-iE_\nu\left(\frac{x}{2}\right)\right]. \tag{46}$$

If we now equate the real and imaginary parts of relations (44) and (46), two different forms of the unique polynomial solution of (36), we obtain the relations

$$(-1)^\nu E_\nu\left(\frac{1-x}{2}\right)=E_\nu\left(\frac{1+x}{2}\right),$$

$$(-1)^\nu E_\nu\left(\frac{2-x}{2}\right)=E_\nu\left(\frac{x}{2}\right),$$

which are not independent but are easily seen to be equivalent to the fundamental identity $E_\nu(1-x)=(-1)^\nu E_\nu(x)$([1]).

Now, let us express $D_\nu(x)$ in still another way, this time in terms of Bernoulli polynomials, the $B_\nu(x)$ of this section. If we expand (39), separate the result into real and imaginary parts, and

([1]) Cf.: Nörlund, *Les Polynomes de Bernoulli*, p. 144.

finally replace x by $\frac{1-x}{n}$, we find that

$$(47) \qquad D_\nu(x)=\frac{n^\nu(-1)^\nu}{\nu+1}\left[\sum_{s=1}^{\frac{n}{2}}(-1)^s B_{\nu+1}\left(\frac{2s-1-x}{n}\right)\right.$$

$$\left.+i\sum_{s=1}^{\frac{n}{2}}(-1)^{s+1}B_{\nu+1}\left(\frac{2s-x}{n}\right)\right],\ n=4k.$$

The particular case of (47), when n is 4, compared with (44) gives

$$E_\nu\left(\frac{1-x}{2}\right)=\frac{2^{\nu+1}}{\nu+1}\left[B_{\nu+1}\left(\frac{3-x}{4}\right)-B_{\nu+1}\left(\frac{1-x}{4}\right)\right],$$

from which readily follows

$$(48) \qquad E_{\nu-1}(x)=\frac{2^\nu}{\nu}\left[B_\nu\left(\frac{x+1}{2}\right)-B_\nu\left(\frac{x}{2}\right)\right]\text{(}^1\text{)}.$$

In the same way, we can derive from (41) the relation

$$(49) \qquad D_\nu(x)=\frac{n^\nu(-1)^\nu}{2}\left[\sum_{s=1}^{\frac{n}{2}}(-1)^{s-1}E_\nu\left(\frac{2s-1-x}{n}\right)\right.$$

$$\left.+i\sum_{s=1}^{\frac{n}{2}}(-1)^s E_\nu\left(\frac{2s-x}{n}\right)\right],\ n=4k+2.$$

Thus, we can see clearly the important rôle the Euler and Bernoulli polynomials play in the polynomial solutions of equations (35) and (36).

Just here it is instructive to note that the solution, $D_\nu(x)$, of equation (36) may be obtained in more direct fashion. In (36), put $f(x)=\alpha(x)+i\beta(x)$. We then have, equating real and imaginary parts,

$$\alpha(x+1)-\beta(x)=x^\nu,$$
$$\beta(x+1)+\alpha(x)=0,$$

which, when combined, give

$$\beta(x+2)+\beta(x)=-x^\nu,$$
$$\alpha(x+2)+\alpha(x)=(x+1)^\nu,$$

whose solutions are, respectively,

$$\beta(x)=-2^{\nu-1}E_\nu\left(\frac{x}{2}\right),$$

(1) This relation is given by Nörlund, *Differenzenrechnung*, p. 24.

$$\alpha(x)=2^{\nu-1}E_\nu\left(\frac{x+1}{2}\right),$$

whence (46) follows. In a similar fashion $C_\nu(x)$ might be found.

5. The solution of $a\nabla bf(x)=x^\nu$, when $-\frac{a}{b}$ is one of the $(n-1)$th roots of unity. It might be noted here that the four equations (34), (35), (36) and (37) are special cases of $af(x+1)+bf(x)=\kappa x^\nu$, where $-\frac{a}{b}$ is one of the fourth roots of unity and κ a constant.

Now let us consider first the difference equation

$$2f(x+1)+(1+i\sqrt{3})f(x)=x^\nu, \tag{50}$$

which is a special case of $a\nabla bf(x)=x^\nu$ with $\frac{a}{a+b}=2$, $\frac{b}{a+b}=1+i\sqrt{3}$ and $-\frac{a}{b}=\frac{-1+i\sqrt{3}}{2}$, one of the three cubic roots of unity. If, in (50), we put $f(x)=\alpha(x)+i\beta(x)$ and then equate real and imaginary parts respectively, we obtain

$$2\alpha(x+1)+\alpha(x)-\sqrt{3}\cdot\beta(x)=x^\nu,$$

$$2\beta(x+1)+\beta(x)+\sqrt{3}\cdot\alpha(x)=0,$$

which, when combined, give

$$\beta(x+2)+\beta(x+1)+\beta(x)=-\frac{\sqrt{3}}{4}x^\nu,$$

$$\alpha(x+2)+\alpha(x+1)+\alpha(x)=\frac{(x+1)^\nu}{2}+\frac{x^\nu}{4},$$

whose solutions are, respectively,

$$\beta(x)=-\frac{3^{\nu+\frac{1}{2}}}{2^2(\nu+1)}\left[B_{\nu+1}\left(\frac{x+1}{3}\right)-B_{\nu+1}\left(\frac{x}{3}\right)\right],$$

$$\alpha(x)=\frac{3^\nu}{2^2(\nu+1)}\left[2B_{\nu+1}\left(\frac{x+2}{3}\right)-B_{\nu+1}\left(\frac{x+1}{3}\right)-B_{\nu+1}\left(\frac{x}{3}\right)\right].$$

Hence the solution of (50) is

$$f(x)=\frac{3^\nu}{2^2(\nu+1)}\left\{2B_{\nu+1}\left(\frac{x+2}{3}\right)-B_{\nu+1}\left(\frac{x+1}{3}\right)-B_{\nu+1}\left(\frac{x}{3}\right)\right.$$
$$\left.-i\sqrt{3}\left[B_{\nu+1}\left(\frac{x+1}{3}\right)-B_{\nu+1}\left(\frac{x}{3}\right)\right]\right\}. \tag{51}$$

We saw, Part 1, 3, (18), that the solution of $a\nabla bU_\nu(x)=x^\nu$ may be expressed in the form $f(x)=n^\nu\sum_{s=0}^{n-1}(-1)^s\left(\frac{a}{b}\right)^sU_\nu\left(\frac{x+s}{n}\right)$, when $-\frac{a}{b}$ is

one of the $(n-1)$th roots of unity.

Now $f(x)$ is readily expressible as the sum of differences Δ^n, $n=1, 2, 3, \ldots, \nu$(1). Let us express $a\nabla b f(x)=x^\nu$ in the symbolic form $\left(\frac{a}{a+b}\Delta+1\right)f(x)=x^\nu$. Hence, since

$$\left(1+\frac{a}{a+b}\Delta\right)\left(1-\left(\frac{a}{a+b}\right)\Delta+\left(\frac{a}{a+b}\right)^2\Delta^2-\ldots.+(-1)^\nu\left(\frac{a}{a+b}\right)^\nu\Delta^\nu\right)x^\nu$$

$$=\left(1+\left(\frac{a}{a+b}\right)^{\nu+1}\Delta^{\nu+1}\right)x^\nu=x^\nu,$$

we see that the solution of the difference equation $a\nabla b f(x)=x^\nu$ is

$$(52)\qquad f(x)=\left(1-\left(\frac{a}{a+b}\right)\Delta+\left(\frac{a}{a+b}\right)^2\Delta^2-\ \ldots+(-1)^\nu\left(\frac{a}{a+b}\right)^\nu\Delta^\nu\right)x^\nu.$$

We can now apply (52) to the case where

$-\frac{a}{b}=\cos\frac{2\pi}{n-1}+i\sin\frac{2\pi}{n-1}=e^{i\frac{2\pi}{n-1}}$, one of the $(n-1)$th roots of unity. We write the difference equation in the form

$$(53)\qquad \theta(x+1)-e^{-i\kappa}\theta(x)=x^\nu,\quad \kappa=\frac{2\pi}{n-1}.$$

Then, using (52) we get

$$(54)\qquad \theta(x)=\frac{1}{1-e^{-i\kappa}}+\left(1-\frac{\Delta}{1-e^{-i\kappa}}+\frac{\Delta^2}{(1-e^{-i\kappa})^2}-\ldots.\right.$$

$$\left.+(-1)^\nu\frac{\Delta^\nu}{(1-e^{-i\kappa})^\nu}\right)x^\nu,$$

and, since $\Delta^m f(x)=\sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}f(x+s)$, we have

$$(55)\qquad \theta(x)=\sum_{m=0}^{\nu}\sum_{s=0}^{m}(-1)^{m-s+1}\left(\frac{e^{i\frac{n+1}{4}\kappa}}{2\sin\frac{\kappa}{2}}\right)^{m+1}\binom{m}{s}(x+s)^\nu,$$

which is the polynomial solution of (53).

Further, let us in (53) put $\theta(x)=g(x)+ih(x)$. Then, equating real and imaginary parts and combining the two resulting equations, we get

$$(56)\qquad g(x+2)-2\cos\kappa\cdot g(x+1)+g(x)=(x+1)^\nu-\cos\kappa\cdot x^\nu,$$

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(1) Cf.: Boole, *A Treatise on the Calculus of Finite Differences*, Chapter XI. Nörlund also defines the Δ^n at the outset in *Les Polynomes de Bernoulli* and in *Differenzenrechnung*.

$$(57)\qquad h(x+2)-2\cos\kappa\cdot h(x+1)+h(x)=-\sin\kappa\cdot x^{\nu}.$$

Now, separating (55) into its real and imaginary parts, we obtain as the polynomial solutions of (56) and (57), respectively,

$$(58)\qquad g(x)=\sum_{n=0}^{\nu}\sum_{s=0}^{m}(-1)^{m-s+1}\frac{\cos(m+1)\frac{n+1}{4}\kappa}{2^{m+1}\sin^{m+1}\frac{\kappa}{2}}\binom{m}{s}(x+s)^{\nu},$$

$$(59)\qquad h(x)=\sum_{m=0}^{\nu}\sum_{s=0}^{m}(-1)^{m-s+1}\frac{\sin(m+1)\frac{n+1}{4}\kappa}{2^{m+1}\sin^{m+1}\frac{\kappa}{2}}\binom{m}{s}(x+s)^{\nu}.$$

6. Properties of certain numbers generated by the difference equation $f\left(x+\frac{1}{2}\right)-if(x)=2x^{\nu}$. A sequence of numbers, closely related to the numbers C_{ν} and E_{ν}, is given by the relation

$$(60)\qquad E_{\nu}\left(x+\frac{1}{2}\right)+iE_{\nu}(x)=ix^{\nu}+\sum_{s=0}^{\nu}\binom{\nu}{s}\frac{A_s}{2^s}\left(x-\frac{1}{2}i\right)^{\nu-s}.$$

Since the left member of (60) satisfies the difference equation

$$(61)\qquad f\left(x+\frac{1}{2}\right)-if(x)=2x^{\nu},$$

then must the right member of (60) satisfy it also. Hence, if we substitute the right member of (60) in (61) and then set $x=\frac{1}{2}i$, we obtain the recursion formula

$$(62)\qquad A_0\left(\frac{1}{2}\right)^{\nu}+\binom{\nu}{1}\frac{A_1}{2}\left(\frac{1}{2}\right)^{\nu-1}+\binom{\nu}{2}\frac{A_2}{2^2}\left(\frac{1}{2}\right)^{\nu-2}+\ldots\ldots+\frac{A_{\nu}}{2^{\nu}}-\frac{A_{\nu}}{2^{\nu}}i$$
$$=\left(\frac{1}{2}i\right)^{\nu}-i\left(\frac{1}{2}i+\frac{1}{2}\right)^{\nu}.$$

The first A's are

$$A_0=1,\ A_1=0,\ A_2=0,\ A_3=i,\ A_4=0,\ A_5=-5i,\ A_6=-10.$$

Now putting $x=0$ in equation (60) and recalling from Part I, 3, closing paragraph, that $E_{\nu}\left(\frac{1}{2}\right)=\frac{E_{\nu}}{2^{\nu}}$ and $E_{\nu}(0)=\frac{C_{\nu}}{2^{\nu}}$, we have

$$(63)\qquad E_{\nu}+iC_{\nu}=(A-i)^{\nu},\ \nu>0.$$

Since $E_{\nu}=0$, when ν is odd, and $C_{\nu}=0$, when ν is even, it follows that

The numbers $E_{\nu}+iC_{\nu}$ and consequently the numbers A_{ν} are either

integers or integral multiples of i.

Let us return to equation (62). If it be written symbolically, then, $\varphi(z)$ being a polynomial in z, we obtain

$$\varphi(A+1)-i\varphi(A)=\varphi(i)-i\varphi(1+i),$$

from which it follows readily that

$$f(x)=\sum_{s=0}^{\nu}\varphi^{(s)}\left(x-\frac{1}{2}i\right)\frac{A^s}{2^s\cdot s!} \tag{64}$$

is a solution of the difference equation

$$f\left(x+\frac{1}{2}\right)-if(x)=\varphi(x)-i\varphi\left(x+\frac{1}{2}\right) \tag{65}$$

and

$$f(x)=\sum_{s=0}^{\nu}\varphi^{(s)}(x-i)\frac{A^s}{s!} \tag{66}$$

is a solution of the difference equation

$$f(x+1)-if(x)=\varphi(x)-i\varphi(x+1). \tag{67}$$

If the right member of (65) be put equal to $2x^{\nu}$, we find

$$f(x)=E_{\nu}\left(x+\frac{A}{2}-\frac{i}{2}\right)+iE_{\nu}\left(x+\frac{A}{2}-\frac{i}{2}+\frac{1}{2}\right). \tag{68}$$

Since the left member of (60) satisfies (61), it follows that

$$E_{\nu}\left(x+\frac{A}{2}-\frac{i}{2}\right)+iE_{\nu}\left(x+\frac{A}{2}-\frac{i}{2}+\frac{1}{2}\right)=E_{\nu}\left(x+\frac{1}{2}\right)+iE_{\nu}(x).$$

If the right member of (67) be put equal to x^{ν}, we find

$$f(x)=iD_{\nu}(x+A-i)=\frac{1}{\nu+1}C_{\nu+1}(x). \tag{68}$$

In similar fashion, we could solve other cases.

Again, let us substitute the right member of equation (60) in equation (61), this time setting $x=0$. We then obtain a summation formula which, expressed symbolically, becomes

$$(A-i+1)^{\nu}-i(A-i)^{\nu}=-i. \tag{70}$$

Putting $A-i=\epsilon$, the relation (70) becomes

$$(\epsilon+1)^{\nu}-i\epsilon^{\nu}=-i,\ \nu=1, 2, 3, \ldots,\ \epsilon_0=1, \tag{71}$$

from which the ϵ's may be computed. The first of them are

$$\epsilon_0=1,\ \epsilon_1=-i,\ \epsilon_2=-1,\ \epsilon_3=2i,\ \epsilon=5,\ \epsilon_5=-16i.$$

Now, since $A-i=\epsilon$, (63) may be written

$$\epsilon_\nu = E_\nu + iC_\nu,\ \nu = 1, 2, 3, \ldots;\ \epsilon_0 = E_0 = C_0 = 1,$$

which is a relation between the numbers ϵ_ν, the Euler numbers and the numbers C_ν.

Part III. The Generalized Euler Polynomial of Order n.

7. The definition of the *generalized Euler polynomial of order* n. In Part I, 1, (5), we have defined the weighted mean of the values of a function $f(x)$ at the points x and $x+\omega$ by the relation $a\underset{\omega}{\nabla} bf(x) = \dfrac{af(x+\omega)+bf(x)}{a+b}$. Now let $\omega_1, \omega_2, \omega_3, \ldots, \omega_n$ be n complex numbers. We then define the weighted mean of the second order by

$$a\overset{2}{\underset{\omega_1,\omega_2}{\nabla}} bf(x) = a\underset{\omega_2}{\nabla} b(a\underset{\omega_1}{\nabla} bf(x))$$
$$= \frac{a^2 f(x+\omega_1+\omega_2) + abf(x+\omega_1) + abf(x+\omega_2) + b^2 f(x)}{(a+b)^2}$$

and in general that of the nth order by

$$a\overset{n}{\underset{\omega_1,\ldots,\omega_n}{\nabla}} bf(x) = a\underset{\omega_n}{\nabla} b\left(a\overset{n-1}{\underset{\omega_1,\ldots,\omega_{n-1}}{\nabla}} bf(x)\right).$$

Consider now the difference equation

$$a\overset{n}{\underset{\omega_1,\ldots,\omega_n}{\nabla}} bf(x) = x^\nu, \tag{72}$$

where $\omega_1, \ldots, \omega_n$ are any complex numbers, n a positive integer, and ν a positive integer or zero. It is evident that there is one and only one polynomial which satisfies (72). It is of degree ν and we shall designate it by $U_\nu^{(n)}(x|\omega_1, \ldots, \omega_n)$ or $U_\nu^{(n)}(x)$. We name it the *generalized Euler polynomial of order* n. We can thus designate the *generalized Euler polynomials* of higher order by $U_\nu^{(1)}(x)$, $U_\nu^{(2)}(x)$, $U_\nu^{(3)}(x)$, ... In particular, since both $U_\nu^{(1)}(x|\omega)$ and $\omega^\nu U_\nu\left(\dfrac{x}{\omega}\right)$ satisfy the equation $a\underset{\omega}{\nabla} bf(x) = x^\nu$, we have $U_\nu^{(1)}(x|\omega) = \omega^\nu U_\nu\left(\dfrac{x}{\omega}\right)$.

8. Properties of the *generalized Euler polynomial of oder* n[1]. From (72), it can readily be shown that

$$a\overset{p}{\underset{\omega_1,\ldots,\omega_p}{\nabla}} bU_\nu^{(n)}(x|\omega_1, \ldots, \omega_n) = U_\nu^{(n-p)}(x|\omega_{p+1}, \ldots, \omega_n). \tag{73}$$

Now, let us put $U_\nu^{(0)}(x) = x^\nu$, thus making equation (73) true for $p=n$. Hence we see from (73) that, the equations

(1) Cf.: Nörlund, *Les Polynomes de Bernoulli*, p. 145. The properties of the $U_\nu^{(n)}(x)$ are closely analogous to those of the $E_\nu^{(n)}(x)$ treated by Nörlund.

$$(74)\quad \begin{cases} a\underset{\omega_n}{\nabla} bU_\nu^{(n)}(x|\omega_1, \ldots\ldots, \omega_n) = U_\nu^{(n-1)}(x|\omega_1, \ldots, \omega_{n-1}), \\ a\underset{\omega_{n-1}}{\nabla} bU_\nu^{(n-1)}(x|\omega_1, \ldots, \omega_{n-1}) = U_\nu^{(n-2)}(x|\omega_1, \ldots, \omega_{n-2}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a\underset{\omega_1}{\nabla} bU_\nu^{(1)}(x|\omega_1) = x^\nu, \end{cases}$$

are equivalent to equation (72).

If we differentiate equation (72) with respect to x, we obtain

$$D_x U_\nu^{(n)}(x) = \nu U_{\nu-1}^{(n)}(x).$$

Then, using Taylor's theorem and equation (73), we have

$$(75)\qquad U_\nu^{(n-1)}(x) = U_\nu^{(n)}(x) + \frac{a}{a+b}\sum_{s=1}^{\nu}\binom{\nu}{s}\omega_n^s U_{\nu-s}^{(n)}(x).$$

Thus the polynomials of order n can be determined successively when those of order $n-1$ are known.

Next, let us consider the difference equation

$$(76)\qquad \underset{\omega_1,\ldots,\omega_p}{a\overset{p}{\nabla} b} f(x) = \varphi(x),$$

$\varphi(x)$ being a polynomial of degree ν. $\varphi(x)$ may be written in the form $A_0 + A_1 x + \ldots + A_\nu x^\nu$. Then we may write

$$f(x) = A_0 U_0^{(p)}(x) + A_1 U_1^{(p)}(x) + \ldots + A_\nu U_\nu^{(p)}(x).$$

Replacing x by $x+y$ and expanding the second member by Taylor's theorem, we find

$$(77)\qquad f(x+y) = \sum_{s=0}^{\nu} \varphi^{(s)}(x)\frac{U_s^{(p)}(y)}{s!}.$$

If we put $\varphi(x) = U_\nu^{(n-p)}(x|\omega_{p+1}, \ldots, \omega_n)$, then $f(x) = U_\nu^{(n)}(x|\omega_1, \ldots \omega_n)$. Finally, substituting in (77), we get

$$(78)\quad U_\nu^{(n)}(x+y|\omega_1, \ldots\ldots \omega_n) = \sum_{s=0}^{\nu}\binom{\nu}{s} U_s^{(p)}(y|\omega_1, \ldots, \omega_p) U_{\nu-s}^{(n-p)}(x|\omega_{p+1}, \ldots, \omega_n).$$

Hence it follows that

The polynomials of order n can be explicitly expressed by means of those of order less than n.

The relation (78) is a direct generalization of the analogous relation

$$E_\nu^{(n)}(x+y|\omega_1, \ldots, \omega_n) = \sum_{s=0}^{\nu}\binom{\nu}{s} E_s^{(p)}(y|\omega_1, \ldots, \omega_p) E_{\nu-s}^{(n-p)}(x|\omega_{p+1}, \ldots, \omega_n),$$

the E's being the values the corresponding U's assume when $a=b=1$. This relation of the E's, given by Nörlund(1), is, on the other hand, more general than (78) in that it subsists if the polynomials $E_s^{(p)}(y)$ $(s=1, 2, \ldots, \nu)$ depend upon any p of the numbers $\omega_1, \omega_2, \ldots, \omega_n$ while the polynomials $E_s^{(n-p)}(x)$ depend upon the $n-p$ remaining numbers. The relation (78) can be expressed in either one of the two symbolic forms,

$$U_\nu^{(n)}(x+y) = U_\nu^{(n-p)}(x+U^{(p)}(y)),$$
$$U_\nu^{(n)}(x+y) = (U^{(n-p)}(x)+U^{(p)}(y))^\nu.$$

We conclude in general that

If $x_1, x_2, \ldots, x_s$ be any numbers and $p_1, p_2, \ldots, p_s$ positive integers such that $p_1+p_2+\ldots+p_s=n$, then

$$U_\nu(x_1+x_2+\ldots+x_s) = [U^{(p_1)}(x_1)+U^{(p_2)}(x_2)+\ldots+U^{(p_s)}(x_s)]^\nu.$$

Now, let us put $p=1$ and $y=0$ in the relation (78). Then we have

$$(79)\quad U_\nu^{(n)}(x) = \sum_{s=0}^{\nu}\binom{\nu}{s}\left(\frac{\omega_1}{a+b}\right)^s K_s U_{\nu-s}^{(n-1)}(x|\omega_2, \omega_3, \ldots, \omega_n),$$

which may also be obtained by solving the relation (75) for $U_\nu^{(n)}(x)$. By means of either (75) or (79) we can determine all the $U_\nu^{(n)}(x)$. Some are

$$U_0^{(n)}(x)=1\,;\quad U_1^{(n)}(x) = x-\frac{a}{a+b}\sum_1^n \omega_s\,;$$

$$U_2^{(n)}(x) = x^2-\frac{2a}{a+b}\sum_1^n \omega_s x+2\left(\frac{a}{a+b}\right)^2\sum \omega_{s_1}\omega_{s_2} + \left[2\left(\frac{a}{a+b}\right)^2-\frac{a}{a+b}\right]\sum_1^n \omega_s^2.$$

At this point we define certain numbers $U_\nu^{(n)}$ and $K_\nu^{(n)}$ respectively, by the symbolic relations

$$(80)\quad a(U^{(n)}+\omega_n)^\nu+b(U^{(n)}-\omega_n)^\nu = (a+b)U_\nu^{(n-1)},\quad U_\nu^{(1)}[\omega_1] = \omega_1^\nu U_\nu$$

and

$$(81)\quad a(K^{(n)}+[a+b]\omega_n)^\nu+b(K^{(n)})^\nu = (a+b)K_\nu^{(n-1)},\quad K_\nu^{(1)}[\omega_1] = \omega_1^\nu K_\nu.$$

(1) Cf.: Nörlund, *Les Polynomes de Bernoulli*, pp. 147-152.

Thus the $U_\nu^{(n)}$ and $K_\nu^{(n)}$ can be determined in a manner analogous to that in which the U_ν and K_ν were determined.

From relations (80) and (81), we have, respectively, when $\varphi(z)$ is a given polynomial,

$$a\varphi(U^{(n)}+\omega_n)+b\varphi(U^{(n)}-\omega_n)=(a+b)\varphi(U^{(n-1)}),$$
$$a\varphi(K^{(n)}+[a+b]\omega_n)+b\varphi(K^{(n)})=(a+b)\varphi(K^{(n-1)}),$$

which, upon replacing $\varphi(z)$ by $\varphi\left(x+\frac{z}{2}\right)$ and $\varphi\left(x+\frac{z}{a+b}\right)$, respectively, become

$$a\varphi\left(x+\frac{U^{(n)}+\omega_n}{2}\right)+b\varphi\left(x+\frac{U^{(n)}-\omega_n}{2}\right)=(a+b)\varphi\left(x+\frac{U^{(n-1)}}{2}\right),$$
$$a\varphi\left(x+\omega_n+\frac{K^{(n)}}{a+b}\right)+b\varphi\left(x+\frac{K^{(n)}}{a+b}\right)=(a+b)\varphi\left(x+\frac{K^{(n-1)}}{a+b}\right).$$

Hence the difference equations

$$af(x+\omega_n)+bf(x)=(a+b)\varphi\left(x+\frac{U^{(n-1)}}{2}\right),$$
$$af(x+\omega_n)+bf(x)=(a+b)\varphi\left(x+\frac{K^{(n-1)}}{a+b}\right)$$

have the solutions

$$f(x)=\varphi\left(x-\frac{\omega_n}{2}+\frac{U^{(n)}}{2}\right), \tag{82}$$

$$f(x)=\varphi\left(x+\frac{K^{(n)}}{a+b}\right) \tag{83}$$

respectively.

Beginning with the last equation of the set (74), we obtain, in connection with (82), after putting $\varphi(x)=x^\nu$ and $\varphi(x)=\left(x-\frac{\omega_1}{2}\right)^\nu$, respectively

$$U_\nu^{(1)}(x\,|\,\omega_1)=\left(x-\frac{\omega_1}{2}+\frac{U^{(1)}}{2}\right)^\nu,$$
$$U_\nu^{(2)}(x\,|\,\omega_1,\,\omega_2)=\left(x-\frac{\omega_1+\omega_2}{2}+\frac{U^{(2)}}{2}\right)^\nu.$$

We conclude generally that

$$U_\nu^{(n)}(x|\omega_1,\,\omega_2,\,\ldots,\,\omega_n)=\left(x-\frac{\omega_1+\omega_2+\ldots+\omega_n}{2}+\frac{U^{(n)}}{2}\right)^\nu. \tag{84}$$

Equation (84) may be put in the form

$$(85)\qquad 2^{\nu}U_{\nu}^{(n)}\left(\frac{x+\omega_1+\omega_2+\ldots.+\omega_n}{2}\right)=\sum_{s=0}^{\nu}\binom{\nu}{s}x_s\,U_{\nu-s}^{(n)}.$$

Similarly, from the set (74) and equation (83) we have putting $\varphi(x)=x^{\nu}$,

$$U_{\nu}^{(1)}(x|\omega_1)=\left(x+\frac{K^{(1)}}{a+b}\right)^{\nu},$$

$$U_{\nu}^{(2)}(x|\omega_1,\,\omega_2)=\left(x+\frac{K^{(2)}}{a+b}\right)^{\nu}.$$

We conclude generally that

$$(86)\qquad U_{\nu}^{(n)}(x|\omega_1,\,\omega_2,\,\ldots.,\,\omega_n)=\left(x+\frac{K^{(n)}}{a+b}\right)^{\nu},$$

or

$$(87)\qquad U_{\nu}^{(n)}(x)=\sum_{s=0}^{\nu}\binom{\nu}{s}\frac{K_s^{(n)}}{(a+b)^s}x^{\nu-s}.$$

Thus we see the important rôle played by $U_{\nu}^{(n)}$ and $K_{\nu}^{(n)}$ in the study of the *generalized Euler polynomial of order n*.

9. Properties of $U_{\nu}^{(n)}$ and $K_{\nu}^{(n)}$ [1]. If we put $x=0$, in equation (85), we obtain

$$(88)\qquad 2^{\nu}U_{\nu}^{(n)}\left(\frac{\omega_1+\omega_2+\omega_3+\ldots.+\omega_n}{2}\right)=U_{\nu}^{(n)}.$$

Also, putting $x=0$ in equation (87), we have

$$(89)\qquad (a+b)^{\nu}U_{\nu}^{(n)}(0)=K_{\nu}^{(n)}.$$

Now in equation (78) put $p=1$ and $x=y=0$. Then we have

$$(90)\qquad K_{\nu}^{(n)}[\omega_1,\,\omega_2,\,\ldots.,\,\omega_n]=\sum_{s=0}^{\nu}\binom{\nu}{s}\omega_1^s K_s K_{\nu-s}^{(n-1)}[\omega_2,\,\omega_3,\,\ldots.,\,\omega_n].$$

Again, if in equation (78) we put $p=1$, $y=\frac{\omega_1}{2}$ and $x=\frac{\omega_2+\omega_3+\ldots.+\omega_n}{2}$, we get

$$(91)\qquad U_{\nu}^{(n)}[\omega_1,\,\omega_2,\,\ldots.,\,\omega_n]=\sum_{s=0}^{\nu}\binom{\nu}{s}\omega_1^s U_s U_{\nu-s}^{(n-1)}[\omega_2,\,\omega_3,\,\ldots.,\,\omega_n].$$

Equations (90) and (91) may be termed, respectively, the inverses of equations (81) and (80). Upon placing $n=2$ in equations (90) and (91), we find, respectively,

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(1) Cf.: Nörlund, *Les Polynomes de Bernoulli*, pp. 152–155.

$$K_\nu^{(2)}[\omega_1, \omega_2]=\sum_{s=0}^{\nu}\binom{\nu}{s}K_s K_{\nu-s}\omega_1^s\omega_2^{\nu-s},$$

$$U_\nu^{(2)}[\omega_1, \omega_2]=\sum_{s=0}^{\nu}\binom{\nu}{s}U_s U_{\nu-s}\omega_1^s\omega_2^{\nu-s}.$$

We have generally

$$K_\nu^{(n)}[\omega_1, \ldots, \omega_n]=\sum\frac{\nu!}{s_1!\, s_2!\, \ldots\, s_n!}K_{s_1}K_{s_2}\ldots K_{s_n}\omega_1^{s_1}\omega_2^{s_2}\ldots\omega_n^{s_n},$$

$$U_\nu^{(n)}[\omega_1, \ldots, \omega_n]=\sum\frac{\nu!}{s_1!\, s_2!\, \ldots\, s_n!}U_{s_1}U_{s_2}\ldots U_{s_n}\omega_1^{s_1}\omega_2^{s_2}\ldots\omega_n^{s_n},$$

where the summation is over all non-negative integral values of s_1, s_2, s_3, $\ldots s_n$, such that $s_1+s_2+s_3+\ldots+s_n=\nu$. Now, if the symbolic relations for $K_\nu^{(n)}$ and $U_\nu^{(n)}$,

$$K_\nu^{(n)}[\omega_1, \omega_2, \ldots, \omega_n]=({}_1K\omega_1+{}_2K\omega_2+\ldots+{}_nK\omega_n)^\nu,$$

$$U_\nu^{(n)}[\omega_1, \omega_2, \ldots, \omega_n]=({}_1U\omega_1+{}_2U\omega_2+\ldots+{}_nU\omega_n)^\nu,$$

be used with equations (86) and (85), respectively, we have

$$(a+b)^\nu U_\nu^{(n)}\left(\frac{x}{a+b}\middle|\omega_1, \ldots, \omega_n\right)=(x+{}_1K\omega_1+{}_2K\omega_2+\ldots+{}_nK\omega_n)^\nu$$

and

$$2^\nu U_\nu^{(n)}\left(\frac{x+\omega_1+\ldots+\omega_n}{2}\middle|\omega_1, \ldots, \omega_n\right)$$
$$=(x+{}_1U\omega_1+{}_2U\omega_2+\ldots+{}_nU\omega_n)^\nu,$$

where, after the multinomial expansion of the second members, $({}_sK)^\nu$ is to be replaced by K_ν and $({}_sU)^\nu$ by U_ν.

The $U_\nu^{(n)}$ and $K_\nu^{(n)}$ are thus symmetric functions of parameters, $\omega_1, \omega_2, \ldots, \omega_n$, as are the polynomials $U_\nu^{(n)}(x)$. These functions are also homogeneous and of degree ν, or

$$K_\nu^{(n)}[\lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n]=\lambda^\nu K_\nu^{(n)}[\omega_1, \omega_2, \ldots, \omega_n],$$

$$U_\nu^{(n)}[\lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n]=\lambda^\nu U_\nu^{(n)}[\omega_1, \omega_2, \ldots, \omega_n],$$

$$U_\nu^{(n)}(\lambda x|\lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n)=\lambda^\nu U_\nu^{(n)}(x|\omega_1, \omega_2, \ldots, \omega_n),$$

λ being any complex number.

Next, let us find relations for $U_\nu^{(n)}$ and $K_\nu^{(n)}$ more general than relations (91) and (90). Upon putting

$$y=\frac{\omega_1+\ldots+\omega_p}{2},\quad x=\frac{\omega_{p+1}+\omega_{p+2}+\ldots+\omega_n}{2}$$

in equation (78) and using equation (88), we get

$$(92)\qquad U_\nu^{(n)}[\omega_1, \ldots, \omega_n] = \sum_{s=0}^{\nu} \binom{\nu}{s} U_s^{(p)}[\omega_1, \ldots, \omega_p] U_{\nu-s}^{(n-p)}[\omega_{p+1}, \ldots, \omega_n],$$

or symbolically,

$$U_\nu^{(n)}[\omega_1, \ldots, \omega_n] = (U^{(p)} + U^{(n-p)})^\nu.$$

Upon placing $y=x=0$ in equation (78), and using equation (89), we have

$$(93)\qquad K_\nu^{(n)}[\omega_1, \ldots, \omega_n] = \sum_{s=0}^{\nu} \binom{\nu}{s} K_s^{(p)}[\omega_1, \ldots, \omega_p] K_{\nu-s}^{(n-p)}[\omega_{p+1}, \ldots, \omega_n],$$

or, symbolically,

$$K_\nu^{(n)}[\omega_1, \ldots, \omega_n] = (K^{(p)} + K^{(n-p)})^\nu.$$

If $p=1$, equations (92) and (93) reduce to equations (91) and (90), respectively.

In general, we have, then

$$(94)\qquad U_\nu^{(n)} = (U^{(p_1)} + U^{(p_2)} + \ldots + U^{(p_s)})^\nu$$

and

$$(95)\qquad K_\nu^{(n)} = (K^{(p_1)} + K^{(p_2)} + \ldots + K^{(p_s)})^\nu,$$

where $p_1, p_2, \ldots, p_s$ are positive integers such that $p_1+p_2+\ldots+p_s=n$. Using equations (94) and (95), $U_\nu^{(n)}$ and $K_\nu^{(n)}$ can be readily determined. Thus, some of their values are

$$U_0^{(n)}=1;\ U_1^{(n)} = -\frac{a-b}{a+b}\sum_1^n \omega_s;\ U_2^{(n)} = \frac{a^2-6ab+b^2}{(a+b)^2}\sum_1^n \omega_s^2 + 2\left(\frac{a-b}{a+b}\right)^2 \sum \omega_{s_1}\omega_{s_2};$$

$$K_0^{(n)}=1;\ K_1^{(n)} = -a\sum_1^n \omega_s;\ K_2^{(n)} = a(a-b)\sum_1^n \omega_s^2 + 2a^2 \sum \omega_{s_1}\omega_{s_2}.$$

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Price per volume (consisting of two parts) payable in advance: 8 yen in Japanese money. Postage inclusive.